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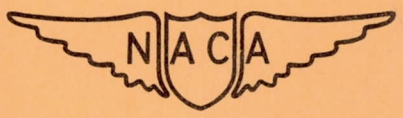

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SUMMARY

An investigation was made to determine the effect of longitudinal stiffeners on the buckling load of long flat plates under shear. Tests were made of long flat plates reinforced by one and by two longitudinal stiffeners. A theoretical study of the buckling load of such plates, made by the energy method, is presented in the appendix. The results of the tests and the results of the theory are compared and are found to be in fair agreement.

INTRODUCTION

Buckling of the stressed skin of a wing under applied shear loads results in a reduced torsional stiffness and a reduced aerodynamic fairness of the wing. Because there is danger of flutter or aileron reversal occurring if the torsional stiffness of the wing is not maintained up to high loads and because high-speed pull-outs may be difficult or impossible if reduced aerodynamic fairness causes premature separation of the flow over the wing, it is desirable to determine the shear stress at which the reinforced skin of the wing buckles. The problem is of particular importance in the case of high-speed airplanes which are normally subject to flutter and control problems. With a view toward eliminating some of the problems in high-speed flight, therefore, a solution to the problem of the shear buckling of a type of panel likely to be used in the wings of fast airplanes has been sought.

The thin wings needed for high-speed airplanes have thick skins and several shear webs. The wing panels are narrow and, therefore, are reinforced by relatively few stiffeners. Accordingly, tests were made to determine the shear buckling load of long plates reinforced by one and by two longitudinal stiffeners. In addition, a theoretical solution of the problem for any number of stiffeners was made. The results of the tests are presented herein and are compared with the results of the theory.

The symbols are defined in the appendix.

TEST SPECIMENS

The specimens tested were flat plates with a length–width ratio of 8 reinforced by longitudinal stiffeners. The general construction of the test specimens is shown in figure 1, and the specific dimensions of the individual specimens are listed in table I.

Two groups of specimens were tested. Each specimen of the first group had one stiffener riveted along the longitudinal center line of the panel. Each specimen of the second group had two longitudinal stiffeners of equal size riveted to the panel in order to divide the panel into three bays of equal width. The dimensions of the specimens of the first group were nominally 6 inches wide and 48 inches long. For the dimensions of specimens of the second group, the width was made $7\frac{7}{8}$ inches in order that the attached legs of the stiffeners would not cover a large part of the width of the bays, and the length was increased to 63 inches in order that, at the same time, the length–width ratio should remain at a value of 8.

The webs of all specimens were of nominally 0.032–inch–thick 24S–T aluminum–alloy sheet. The stiffeners were of 24S–T aluminum–alloy sheet bent to the shape of angles. Two angles were used for each stiffener, one on each side of the web, to provide symmetry about the plane of the web. The thickness and leg dimensions of the angles were varied to produce the bending stiffness desired.

The short edges (ends) of the specimens were reinforced with angles. These angles were of uniform size for all specimens with one stiffener and were proportionally larger and of uniform size for all specimens with two stiffeners. These end angles were so designed that there was a margin of safety against failure of the angles before buckling of the web occurred.

TEST APPARATUS AND TESTING PROCEDURE

The specimens were tested in a jig as shown in figure 2. One part of this jig distributed the applied load along one edge of the web, and the other part picked up the reaction from the opposite edge of the web and transferred this reaction to a heavy supporting structure. Both parts of the jig were essentially the same, and each part consisted of two heavy steel bars bolted to each side of a steel plate which protruded from between the bars and to which the specimen was riveted. The large cross–sectional area of the bars insured that the distribution of load was essentially uniform over the full length of the web.

A portable hydraulic jack which indicated loads with standard testing–machine accuracy of one–half of 1 percent, was used to apply the load to the specimens.

Two dial gages graduated to 1/10000 inch were used to measure the shear displacement of the loaded edge of the sheet relative to the fixed

edge. These gages were mounted on each side of the web at the midlength of the specimen.

The test procedure was as follows: The specimen was preloaded in several increments to about 25 percent of the estimated buckling load, and the dial gages were read after each increment of load had been applied. If the dial-gage readings indicated equal movement on both sides of the web up to the full preload, it was assumed that the jack was properly positioned under the specimen and the load was released. The load was reapplied and the dial-gage readings were taken at a number of loads until buckles were clearly visible in the web. Readings were then taken less frequently until the specimen could sustain no further increase in load.

ANALYSIS AND DISCUSSION

Theoretical critical stresses.- The theoretical study of the shear buckling load of a long flat plate, reinforced by longitudinal stiffeners, with shear load acting at the longitudinal edges is presented in the appendix. Two different restraining conditions at the longitudinal edges are investigated - simply supported edges and clamped edges. The results of the theoretical study are summarized in figure 3. For each edge restraint condition, three separate curves are shown. The curves correspond to the plate reinforced by one stiffener, by two identical equally spaced stiffeners, and by a large number of identical equally spaced stiffeners. The three curves are essentially one with a maximum deviation of approximately 2 or 3 percent. The appendix points out that the curve for a large number of identical equally spaced stiffeners can be used to represent, within 2 or 3 percent, the solution of the plate reinforced by any number of identical equally spaced stiffeners.

Each curve in figure 3 has an upper limit corresponding to the criterion that the plate buckles in such a way that the stiffeners can be replaced by simple supports. In the case of a simply supported plate with one stiffener, for instance, the shear buckling coefficient k cannot be increased beyond approximately 21.4 by increasing the stiffness of the stiffener.

Experimental buckling data.- In figure 4 are shown typical results of a test in the form of a curve of shear deformation, as measured by the dial gages, plotted against load on the specimen. The first part of this curve is linear (i.e., deformation is proportional to load) and corresponds to a constant shear stiffness for the web. The second part above the linear part shows a gradual increase in the rate of deformation of the web with load (i.e., a gradual decrease in the shear stiffness of the web). Since, for all specimens tested, the stress at which the second part of the curve started was well below the yield stress for the web material in shear, it is reasonable to assume that the change in shear stiffness of the web was due to buckling. Since, however, the shear stiffness changes very gradually, it is difficult to select consistent buckling loads from plots such as figure 4; and in order that the selection be confined to a

reasonably short range of load, the effect of a change in the shear stiffness of the web was accentuated in the following manner.

Method for defining experimental critical stress.– For each specimen tested, the deviation of the load–deformation curve from an extension of the linear part of that curve was computed. This quantity was squared and plotted against the load on the specimen. The resulting plots of load against deviation squared, arranged in the order of increasing stiffness ratio γ, are shown in figure 5(a) for one–stiffener specimens and in figure 5(b) for two–stiffener specimens. Each of the curves of the figure exhibits a knee which starts as soon as the deformation of the specimen is no longer proportional to the load. The buckling load corresponds to some point on the knee of the curve of load against deviation squared.

Comparison of theory and experiment.– A theoretical critical buckling load P_{cr}, based on the buckling coefficient from figure 3 and on the dimensions listed in table 1, is marked on each test curve of figures 5(a) and 5(b) in order to provide a means of direct comparison between test data and theory. The subscripts s and c are used to denote whether the edges are simply supported or clamped. It will be noted that for 18 of the 20 specimens the start of the knee of the test curve lies within or very close to one edge of the range bracketed by the two extreme values $\left(P_{cr}\right)_s$ and $\left(P_{cr}\right)_c$.

Maximum stresses.– The maximum load sustained by each specimen is marked on the curves of figures 5(a) and 5(b). For both one–stiffener and two–stiffener groups of specimens, the load at failure tended to increase slightly with increase in the stiffness ratio γ. In all cases, failure ultimately occurred by twisting and collapse of the angle across the top edge of the specimen. Since this top angle was of one size for all one–stiffener specimens and of another size for all two–stiffener specimens, the size and proportions of the longitudinal stiffeners must have affected somewhat the maximum load carried by the specimen by restraining the top angle from twisting.

CONCLUDING REMARKS

The theoretical shear buckling coefficient for a long flat plate reinforced with any number of longitudinal stiffeners, of equal stiffness and equally spaced across the plate, can be obtained from a single curve for each of the edge conditions – simply supported or clamped. The test results were found to be in fair agreement with the theoretical studies.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., December 4, 1947

APPENDIX

THEORETICAL ANALYSIS

Two solutions applicable to the problem of the shear buckling of a flat plate reinforced by longitudinal stiffeners are presented in references 1 and 2. The first of these two solutions (reference 1, p. 360) is an approximate solution for the case of simply supported edges and is obtained by the energy method. The deflection function used was limited to a single half sine wave across the width of the plate and did not completely satisfy the conditions of simply supported edges along the length of the plate. The solution consequently yields buckling loads which are too high and unconservative. The second of the two solutions (reference 2) presents results for the shear buckling of a long orthogonal-anisotropic (often called orthotropic) flat plate with either simply supported or clamped edges. These results were obtained by solving the differential equations of equilibrium of a slightly deflected plate element. It is reasonable to expect that this second solution would be applicable to the case of a plate reinforced by numerous closely spaced and uniformly spaced longitudinal stiffeners.

It was deemed desirable to obtain a more exact solution than given in reference 1 for a long flat plate reinforced by only a few longitudinal stiffeners and to obtain some idea of the extent to which the solution for an orthotropic plate in reference 2 is applicable to a plate with a finite number of longitudinal stiffeners. Two energy solutions were therefore obtained. The first solution was for a plate with a few or a finite number of stiffeners and the second solution, for a plate with a very large number of identical closely spaced stiffeners. (In both cases the stiffeners were assumed to have some flexural stiffness but zero torsional stiffness.)

Two different edge conditions were investigated. In the case of simply supported edges, the infinite series type of deflection function introduced by Kromm (reference 3) was used. This function not only provides simple support along the edges but also provides a complete set of functions which describe the shape of the deflected surface at any section across the plate. Also, in the case of clamped edges, a complete set was used. With either function, it is possible to approach as closely as desired the exact solution to the problem.

SYMBOLS

b	width of plate
c_i	distance from x-axis to ith stiffener
d	stiffener spacing
i, j	stiffener under consideration
k	shear buckling coefficient $\left(\frac{(F_{xy})_{cr} b^2}{\pi^2 D}\right)$
m, n, p	integral number of half waves across plate
t	thickness of plate
w	deflection in z-direction
w_p	deflection of plate in z-direction
w_s	deflection of stiffener in z-direction
x, y, z	coordinate axes
A_m, B_m A_n, B_n A_p, B_p	parameters used in deflection function of plate
D	flexural stiffness per unit width of plate $\left(\frac{E_p t^3}{12(1-\mu^2)}\right)$
E_p	Young's modulus for plate material
EI	bending flexibility of stiffener
$E_i I_i$	bending flexibility of the ith stiffener
F_{xy}	resultant shear force per unit length acting in middle plane of plate
J	number of bays across plate
N	number of stiffeners

T work done by applied shear

V_p energy stored in buckled plate

V_s energy stored in bent stiffeners

β ratio of half wave length in x-direction to width of plate $\left(\frac{\lambda}{b}\right)$

γ ratio of stiffness of single stiffener to stiffness of a strip of plate of width d when all ribs are identical and equally spaced $\left(\frac{EI}{Dd}\right)$

γ_i ratio of stiffness of ith stiffener to stiffness of plate $\left(\frac{E_i I_i}{Db}\right)$

δ_{0n} Kronecker delta (1 if $n = 0$; 0 if $n \neq 0$)

δ^J_{mp} symbol representing the sum of a trigonometric series which takes the value 1 if $\frac{m-p}{J}$ is even and if $\frac{m+p}{J}$ is not even, -1 if $\frac{m+p}{J}$ is even and if $\frac{m-p}{J}$ is not even, and 0 for all other cases

Δ_i, Γ_i parameters used in deflection function of ith stiffener

λ half wave length in x-direction

μ Poisson's ratio for plate material (taken as 0.3)

$\nu, \epsilon, \phi_i, \psi_i$ Lagrangian multipliers

$$F = \frac{D}{2}\left(\frac{\pi}{\lambda}\right)^4 \frac{\lambda}{2} \frac{b}{2}$$

$$G = \sum_{m=1}^{\infty} m^2 R_m$$

$$H_i = \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^m R_m \cos \frac{2m\pi c_i}{b}$$

$$K_m = \left(1 + m^2\beta^2\right)^2$$

$$K_m' = \left(1 + 4m^2\beta^2\right)^2$$

$$L_i = \sum_{m=1}^{\infty} m(-1)^m R_m \sin \frac{2m\pi c_i}{b}$$

$$M_{mp} = \frac{8}{\pi} k\beta^3 \left(\frac{mp}{m^2 - p^2}\right) = 0 \quad \text{when} \quad m \pm p \quad \text{is even}$$

$$N_{mp} = \sum_{i=1,2,3,\ldots}^{N} \gamma_i \sin \frac{m\pi c_i}{b} \sin \frac{p\pi c_i}{b}$$

$$P_i = \sum_{m=1}^{\infty} (-1)^m S_m \sin \frac{2m\pi c_i}{b}$$

$$Q = \frac{\left(F_{xy}\right)_{cr} \pi^2}{2F} = 4\beta^3 k$$

$$R_m = \frac{K_m'}{\left(K_m'\right)^2 - (mQ)^2}$$

$$R_m' = \frac{K_m + \gamma}{\left(K_m + \gamma\right)^2 - (mQ)^2}$$

$$R = \frac{1}{2} + \sum_{m=1}^{\infty} R_m$$

$$S_m = \frac{mQ}{\left(K_m'\right)^2 - (mQ)^2}$$

$$S_m' = \frac{mQ}{\left(K_m + \gamma\right)^2 - (mQ)^2}$$

$$S = \sum_{m=1}^{\infty} mS_m$$

$$T_i = \sum_{m=1}^{\infty} m(-1)^m S_m \cos \frac{2m\pi c_i}{b}$$

$$W_{ij} = \sum_{m=1}^{\infty} S_m \cos \frac{2m\pi c_i}{b} \sin \frac{2m\pi c_j}{b}$$

$$X_{ij} = \sum_{m=1}^{\infty} R_m \sin \frac{2m\pi c_i}{b} \sin \frac{2m\pi c_j}{b}$$

$$Z_{ij} = \frac{1}{2} + \sum_{m=1}^{\infty} R_m \cos \frac{2m\pi c_i}{b} \cos \frac{2m\pi c_j}{b}$$

$$\gamma_o = \frac{\gamma}{J}$$

$$\nu' = \frac{\nu}{2F}$$

$$\epsilon' = \frac{\epsilon}{2F}$$

$$\phi_i' = \frac{\phi_i}{2F}$$

$$\psi_i' = \frac{\psi_i}{2F}$$

Subscript:

cr critical

SIMPLY SUPPORTED EDGES

Finite Number of Stiffeners

The deflection function

$$w = \sin\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} A_m \sin\frac{m\pi y}{b} + \cos\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} B_m \sin\frac{m\pi y}{b} \tag{1}$$

is assumed to express the buckled shape of the plate. (The coordinate system is shown in fig. 6.) The integer m in the deflection function represents the number of half waves across the plate, and the parameters A_m and B_m are associated with the amplitude of the mth wave. The width of the plate is b and the half wave length of the buckle in the x-direction is λ.

In order to find the energy stored in the buckled plate V_p, the energy stored in the bent stiffeners V_s, and the work T done by the applied shear, the following equations were used:

$$V_p = \frac{D}{2}\int_0^b\int_0^\lambda \left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy \tag{2a}$$

which is equation (199) of reference 1, and where D is the flexural stiffness per unit width of plate;

$$V_s = \frac{1}{2}\sum_{i=1}^{N} E_i I_i \int_0^\lambda \left(\frac{\partial^2 w}{\partial x^2}\right)^2_{y=c_i} dx \tag{2b}$$

where $E_i I_i$ is the flexural rigidity of the ith stiffener and c_i is the distance of the ith stiffener from the edge of the plate (only bending energy of the stiffeners is considered and the summation is extended over all the stiffeners on the plate); and

$$T = -F_{xy}\int_0^b \int_0^\lambda \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dx\,dy \qquad (2c)$$

which is obtained from equation (201) of reference 1, and where F_{xy} is the resultant shear force per unit length acting along the longitudinal edge of the plate.

It was necessary to consider only the energy and work over one half wave length along the length of the plate since the variation in deflection is sinusoidal in that direction.

The deflection function was substituted in equations (2) for energy and work, and the indicated integrations were performed with the following results:

$$V_p = \frac{D}{2}\frac{\pi^4 b}{4\lambda^3}\sum_{m=1}^{\infty}\left(A_m^2 + B_m^2\right)\left(1 + \frac{m^2\lambda^2}{b^2}\right)^2 \qquad (3a)$$

$$V_s = \frac{\lambda}{2}\frac{\pi^4}{\lambda^4}\sum_{i=1}^{N}\frac{E_i I_i}{2}\left[\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\left(A_m A_p + B_m B_p\right)\sin\frac{m\pi c_i}{b}\sin\frac{p\pi c_i}{b}\right] \qquad (3b)$$

$$T = -2F_{xy}\pi\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}A_m B_p\frac{mp}{m^2 - p^2} \qquad (3c)$$

where a value for T exists only when $m \pm p$ is odd.

When the buckling load is reached, the structure is in neutral equilibrium and is capable of maintaining either the flat or buckled form. Mathematically, this neutral equilibrium can be expressed by setting the work done by the external load in deforming the plate equal to the sum of the energies stored in the buckled plate and in the bent stiffeners; that is,

$$T = V_p + V_s$$

From this equation, the following critical shear force per unit length of the plate is obtained:

$$\left(F_{xy}\right)_{cr} = -\frac{\pi^2 D}{b^2}\frac{\pi}{8}\frac{b^3}{\lambda^3}\frac{1}{2\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}A_m B_p \frac{mp}{m^2-p^2}}\left[\sum_{m=1}^{\infty}\left(A_m^2 + B_m^2\right)\left(1+\frac{m^2\lambda^2}{b^2}\right)^2\right.$$

$$\left. + \sum_{i=1}^{N} 2\frac{E_i I_i}{Db}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\left(A_m A_p + B_m B_p\right)\sin\frac{m\pi c_i}{b}\sin\frac{p\pi c_i}{b}\right] \qquad (4a)$$

Considerable simplification in the form of equation (4a) can be made by use of the following substitutions:

$$\beta = \frac{\lambda}{b}$$

$$\gamma_i = \frac{E_i I_i}{Db}$$

$$k = \frac{\left(F_{xy}\right)_{cr} b^2}{\pi^2 D}$$

$$K_m = \left(1 + m^2\beta^2\right)^2$$

$$N_{mp} = 2\sum_{i=1}^{N}\gamma_i \sin\frac{m\pi c_i}{b}\sin\frac{p\pi c_i}{b} = N_{pm}$$

Equation (4a) now reduces to

$$k = -\frac{\pi}{8\beta^3}\frac{\sum_{m=1}^{\infty}\left(A_m^2 + B_m^2\right)K_m + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\left(A_m A_p + B_m B_p\right)N_{mp}}{2\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}A_m B_p \frac{mp}{m^2-p^2}} \qquad (4b)$$

The wave pattern which causes the shear buckling coefficient k to be a minimum is obtained by differentiating equation (4b) with respect to each of the parameters A_m and B_m in turn and setting each of the derivatives equal to zero. Two sets of linear algebraic equations in A_m and B_m result. Thus

$$A_m K_m + \sum_{p=1}^{\infty} \left(N_{mp} A_p + M_{mp} B_p \right) = 0 \tag{5a}$$

$$B_m K_m + \sum_{p=1}^{\infty} \left(N_{mp} B_p - M_{mp} A_p \right) = 0 \tag{5b}$$

where

$$M_{mp} = \frac{8k\beta^3}{\pi} \frac{mp}{m^2 - p^2} = -M_{pm}$$

and $M_{mp} = 0$ when $m \pm p$ is even.

Neither equation (5a) nor equation (5b) contains a constant term; therefore, in order that the parameters A_m and B_m have values different from zero (i.e.; the plate takes a form other than flat), the determinant in the coefficients of the A_m's and B_m's must be equal to zero. The complete determinant is infinite in extent and may be represented in the following manner:

	A_1	B_2	A_3	B_4	. . .	B_1	A_2	B_3	A_4	
Equation (5a), m=1	K_1+N_{11}	M_{12}	N_{13}	M_{14}	. . .	0	N_{12}	0	N_{14}	. . .
Equation (5b), m=2	$-M_{21}$	K_2+N_{22}	$-M_{23}$	N_{24}	. . .	N_{21}	0	N_{23}	0	. . .
Equation (5a), m=3	N_{31}	M_{32}	K_3+N_{33}	M_{34}	. . .	0	N_{32}	0	N_{34}	. . .
Equation (5b), m=4	$-M_{41}$	N_{42}	$-M_{43}$	K_4+N_{44}	. . .	N_{41}	0	N_{43}	0	. . .
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
Equation (5b), m=1	0	N_{12}	0	N_{14}	. . .	K_1+N_{11}	$-M_{12}$	N_{13}	$-M_{14}$	. . .
Equation (5a), m=2	N_{21}	0	N_{23}	0	. . .	M_{21}	K_2+N_{22}	M_{23}	N_{24}	. . .
Equation (5b), m=3	0	N_{32}	0	N_{34}	. . .	N_{31}	$-M_{32}$	K_3+N_{33}	$-M_{34}$	. . .
Equation (5a), m=4	N_{41}	0	N_{43}	0	. . .	M_{41}	N_{42}	M_{43}	K_4+N_{44}	. . .
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.

$= 0$ (5c)

General analysis of determinant.- Determinant (5c) may be factored as follows: First, multiply all the odd columns in the right half of the determinant by $\sqrt{-1}$ and all the even columns in the right half of the determinant by $-\sqrt{-1}$, and add each column to the corresponding column in the left half of the determinant; and, second, multiply the odd rows in the upper half of the resulting determinant by $-\sqrt{-1}$ and the even rows in the upper half by $\sqrt{-1}$, and add each row to the

corresponding row in the lower half of the determinant. The lower left quadrant of this altered determinant contains only zeros; the upper left and lower right quadrants are equivalent and are factors of the original determinant. The factored determinant (infinite in extent) can be represented as follows:

$$\begin{vmatrix} K_1 + N_{11} & M_{12} - \sqrt{-1}\,N_{12} & N_{13} & M_{14} - \sqrt{-1}\,N_{14} & \cdots \\ M_{12} + \sqrt{-1}\,N_{12} & K_2 + N_{22} & -M_{23} + \sqrt{-1}\,N_{23} & N_{24} & \cdots \\ N_{13} & -M_{23} - \sqrt{-1}\,N_{23} & K_3 + N_{33} & M_{34} - \sqrt{-1}\,N_{34} & \cdots \\ M_{14} + \sqrt{-1}\,N_{14} & N_{24} & M_{34} + \sqrt{-1}\,N_{34} & K_4 + N_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0 \qquad (5d)$$

It is not readily apparent how the factored determinant might yield a solution in series form; but in order to obtain approximations to the true answer for the problem, finite subdeterminants can be used.

A first approximation is obtained by considering only the terms common to the first two rows and first two columns of determinant (5d) – the determinant which results from summing m and p in equations (5a) and (5b) over 1 and 2. Such a procedure is equivalent to the use of a limited deflection function, which is a combination of only one and two half waves across the plate. A second approximation is obtained by summing m and p over 1, 2, and 3, which is equivalent to adding to the deflection function used for the first approximation, a term containing three half waves across the plate. Similarly, approximations of higher order are obtained by adding to the deflection function, terms containing more half waves across the plate.

For example, from determinant (5d) there is obtained for the first approximation the following subdeterminant:

$$\begin{vmatrix} K_1 + N_{11} & M_{12} - \sqrt{-1}\, N_{12} \\ M_{12} + \sqrt{-1}\, N_{12} & K_2 + N_{22} \end{vmatrix} = 0 \qquad (5e)$$

The expansion of determinant (5e) is

$$\left(K_1 + N_{11}\right)\left(K_2 + N_{22}\right) - N_{12}^2 - M_{12}^2 = 0 \qquad (6)$$

Equation (6) represents a stability criterion for the shear buckling of a long plate with any number, spacing, and size of stiffeners along the plate. If the number, spacing, and size of stiffeners are known, the proper values of N_{mp} may be substituted in equation (6) and the resulting equation solved for k or γ.

Special analysis of determinant for identical equally spaced stiffeners.- Generally, the stiffeners on a plate are of uniform size and are uniformly spaced. Some simplification in the foregoing solution may then be introduced.

In general,

$$N_{mp} = 2 \sum_{i=1}^{N} \gamma_i \sin \frac{m\pi c_i}{b} \sin \frac{p\pi c_i}{b}$$

If the stiffness ratio γ_i is the same for all stiffeners and J is the number of bays across the plate, then

$$N_{mp} = 2\gamma_i \sum_{i=0}^{J} \sin \frac{m\pi}{b} i \frac{b}{J} \sin \frac{p\pi}{b} i \frac{b}{J}$$

$$= 2\frac{EI}{Db}\frac{J}{2}\delta^J{}_{mp}$$

$$= \frac{EI}{Dd}\delta^J{}_{mp}$$

$$= \gamma\delta^J{}_{mp}$$

where

$d = \frac{b}{J}$

$\delta^J_{mp} = 1$ if $\frac{m - p}{J}$ is even and if $\frac{m + p}{J}$ is not even

$\delta^J_{mp} = -1$ if $\frac{m + p}{J}$ is even and if $\frac{m - p}{J}$ is not even

$\delta^J_{mp} = 0$ in all other cases

(See reference 4 for derivation of this sum.)

The determinant (5c) may now be rewritten in terms of δ^J_{mp} rather than N_{mp}. The upper right and lower left quadrants will be composed entirely of zeros since δ^J_{mp} has no value if $m \pm p$ is odd. The upper left and lower right quadrants then remain as equivalent factors. The rewritten upper left quadrant (one factor) of determinant (5c) appears as follows:

$$\begin{vmatrix} K_1 + \gamma\delta^J_{11} & M_{12} & \gamma\delta^J_{13} & M_{14} & \cdots \\ M_{12} & K_2 + \gamma\delta^J_{22} & -M_{23} & \gamma\delta^J_{24} & \cdots \\ \gamma\delta^J_{13} & -M_{23} & K_3 + \gamma\delta^J_{33} & M_{34} & \cdots \\ M_{14} & \gamma\delta^J_{24} & M_{34} & K_4 + \gamma\delta^J_{44} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \qquad (7)$$

Since the M_{mp}-terms containing the shear buckling coefficient k appear in each column, the order in k of the expansion of determinant (7) is the same as the order of the determinant itself.

A study of δ^J_{mp} and determinant (7) shows that the value of δ^J_{mp} alternates and recurs at intervals and that all the values

of δ^J_{mp} are 0 when either m or p is equal to J. Adding and subtracting appropriate rows and columns eliminates the δ^J_{mp}-terms (also γ) from all but the first $J-1$ columns. As a result, the order in γ of the expression resulting from the expansion of determinant (7) is $J-1$. It is therefore more expedient to solve this expression for γ than for k.

As in the case of the more general determinant (5c), the finite subdeterminants of determinant (7) can be used to obtain approximations to the true answer of the problem. Criterions resulting from first, second, and third approximations are given as follows for the web with one stiffener ($J = 2$) and with two stiffeners ($J = 3$):

First approximation, $J = 2$

$$\gamma = K_1\left(\frac{M_{12}^2}{K_1K_2} - 1\right) \tag{8a}$$

Second approximation, $J = 2$

$$\gamma = \frac{\frac{M_{23}^2}{K_2K_3} + \frac{M_{12}^2}{K_1K_2} - 1}{\frac{1}{K_1} + \frac{1}{K_3} - \frac{(M_{23} - M_{12})^2}{K_1K_2K_3}} \tag{8b}$$

Third approximation, $J = 2$

$$\gamma = \frac{\frac{M_{34}^2}{K_3K_4} + \frac{M_{14}^2}{K_1K_4} + \frac{M_{23}^2}{K_2K_3} + \frac{M_{12}^2}{K_1K_2} - \frac{(M_{12}M_{34} + M_{14}M_{23})^2}{K_1K_2K_3K_4} - 1}{\frac{1}{K_1} + \frac{1}{K_3} - \frac{(M_{34} + M_{14})^2}{K_1K_3K_4} - \frac{(M_{23} - M_{12})^2}{K_1K_2K_3}} \tag{8c}$$

First approximation, J = 3

$$\gamma = \frac{1}{2}\left[\sqrt{(K_1 - K_2)^2 + 4M_{12}^2} - (K_1 + K_2)\right] \tag{9a}$$

Second approximation, J = 3

$$\gamma = \frac{1}{2}\left[\sqrt{\left(K_1 - K_2 + \frac{M_{23}^2}{K_3}\right)^2 + 4M_{12}^2} - \left(K_1 + K_2 - \frac{M_{23}^2}{K_3}\right)\right] \tag{9b}$$

Third approximation, J = 3

$$\gamma = \frac{1}{2}\left[\frac{1}{K_2 + K_4 - \frac{1}{K_3}(M_{23} + M_{34})^2}\right]\left[\left(\left\{K_1K_2 + K_1K_4 - K_2K_4 - \left[\frac{K_1}{K_3}(M_{23} + M_{34})^2 - \frac{K_2}{K_3}M_{34}^2 - \frac{K_4}{K_3}M_{23}^2\right] - (M_{12} - M_{14})^2\right\}^2 + 4\left[K_2M_{14} + K_4M_{12} - \frac{1}{K_3}(M_{23} + M_{34})(M_{12}M_{34} + M_{14}M_{23})\right]\right)^{1/2} - \left\{K_1K_2 + K_1K_4 + K_2K_4 - \left[\frac{K_1}{K_3}(M_{23} + M_{34})^2 + \frac{K_2}{K_3}M_{34}^2 + \frac{K_4}{K_3}M_{23}^2\right] - (M_{12} - M_{14})^2\right\}\right] \tag{9c}$$

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In equations (8) to (9) the value of γ is a function of k and β. For any of these equations, γ can be computed for various values of β at a prescribed value of k. The maximum value of γ for the prescribed k can be obtained from a plot of γ against β. Curves of k plotted against γ, as shown in figures 3 and 7, are obtained by repeating the procedure for other values of k. It appears from figure 7 that the third-approximation solution is accurate enough and the solution of the determinant of higher orders is not needed.

Large Number of Stiffeners

If all the stiffeners are identical and are equally spaced, the condition of uniform longitudinal stiffness at any point across the width of the plate is approached as the number of stiffeners is increased. Consequently, for a very large number of stiffeners, the bending energy stored in all the stiffeners may be expressed as

$$V_s = \frac{1}{2}\frac{EI}{d}\int_0^b \int_0^\lambda \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\, dy$$

The work done by the applied shear T and the energy stored in the plate V_p are given by the same expressions used in the development for a finite number of stiffeners. Following the same procedure as that used for the case of a finite number of stiffeners – that is, equating energy and work, solving for the critical load, setting the derivatives with respect to each of the unknown parameters A_m and B_m equal to zero, setting a finite determinant in these parameters equal to zero, and solving in this case for the shear buckling coefficient k – results in the following equation:

$$k^3 = \frac{(K_1 + \gamma)(K_2 + \gamma)(K_3 + \gamma)}{M_{12}^2(K_3 + \gamma) + M_{23}^2(K_1 + \gamma)} \qquad (10)$$

Equation (10) was obtained by a second approximation, that is, by limiting the deflection function to one, two, and three half waves across the width of the plate. By a graphical procedure, as previously presented, there is found for each value of γ, chosen in equation (10), a value of β which will make k a minimum. A curve of k plotted against γ obtained from equation (10) is shown in figure 8. For

purposes of comparison, there is also plotted in the same figure the results of the exact solution for an orthogonal plate from reference 2. The second approximation is sufficient to give excellent agreement with results obtained from the exact solution.

Although equation (10) is obtained for the case of a very large number of stiffeners, it actually represents the second-approximation solution for all cases of three or more stiffeners. This fact can be seen from equation (7) in that the second-approximation solution of equation (7) for all cases of three or more stiffeners gives the same results as equation (10).

CLAMPED EDGES

Finite Number of Stiffeners

The problem is more involved in the case of clamped edges. A different deflection function is assumed and the Lagrangian multiplier method (references 5 and 6) is used. With the new coordinate system shown in figure 9, the deflection function for the plate is assumed to be expressed by

$$w_p = \sin \frac{\pi x}{\lambda} \sum_{m=1}^{\infty} A_m \sin \frac{2m\pi y}{b} + \cos \frac{\pi x}{\lambda} \sum_{n=0}^{\infty} B_n \cos \frac{2n\pi y}{b} \qquad (11)$$

This expression is a complete set of functions symmetric with respect to the origin. Since the plate is infinitely long, the expression of deflection by a complete set of antisymmetrical functions will give the same results. In order to ensure zero deflection and zero slope at the edges, the expression w_p (equation (11)) is subject to the following restraining conditions:

$$\sum_{n=0}^{\infty} (-1)^n B_n = 0 \qquad (12a)$$

$$\sum_{m=1}^{\infty} m(-1)^m A_m = 0 \qquad (12b)$$

A set of new deflection functions is used for the longitudinal stiffeners. For the ith stiffener,

$$\left(w_s\right)_i = \Delta_i \sin \frac{\pi x}{\lambda} + \Gamma_i \cos \frac{\pi x}{\lambda} \qquad (i = 1,2,\ldots N) \tag{13}$$

where N is number of stiffeners. In order that the deflection of the plate directly under the stiffener and the deflection of the stiffener itself be the same, the expressions for w_p and $\left(w_s\right)_i$ are subject to the following restraining conditions:

$$\Delta_i - \sum_{m=1}^{\infty} A_m \sin \frac{2m\pi c_i}{b} = 0 \qquad (i = 1,2,\ldots N) \tag{14a}$$

$$\Gamma_i - \sum_{n=0}^{\infty} B_n \cos \frac{2n\pi c_i}{b} = 0 \qquad (i = 1,2,\ldots N) \tag{14b}$$

where c_i is the distance from the x-axis to the ith stiffener.

When the expressions for w_p and w_s are substituted in equations (2), the energy expressions become

$$V_p = F\left[\sum_{m=1}^{\infty} K_m' A_m^2 + \sum_{n=0}^{\infty} K_n' B_n^2 \left(1 + \delta_{0n}\right)\right] \tag{15a}$$

$$V_s = 2F \sum_{i=1}^{N} \gamma_i \left(\Delta_i^2 + \Gamma_i^2\right) \tag{15b}$$

$$T = F_{xy} \pi^2 \sum_{m=1}^{\infty} m A_m B_m \tag{15c}$$

where

$$F = \frac{D}{2}\left(\frac{\pi}{\lambda}\right)^4 \frac{\lambda}{2}\frac{b}{2}$$

$$K_m' = \left(1 + 4m^2\beta^2\right)^2$$

$$\beta = \frac{\lambda}{b}$$

$$\gamma_i = \frac{E_i I_i}{Dd}$$

$$\delta_{On} = 1 \quad \text{if} \quad n = 0$$

$$\delta_{On} = 0 \quad \text{if} \quad n \neq 0$$

The total energy $V = V_p + V_s - T$ is then to be minimized. During the minimizing process the restraining conditions, equations (12) and (14), can be satisfied by the Lagrangian multiplier method. (See reference 5.)

The following notation is used for the Lagrangian multipliers: ν and ϵ correspond to equations (12a) and (12b), respectively; ϕ_i and ψ_i correspond to equations (14a) and (14b), respectively. Then the function to be minimized is

$$f = V_p + V_s - T - \nu\sum_{n=0}^{\infty}(-1)^n B_n - \epsilon\sum_{m=1}^{\infty}(-1)^m m A_m$$

$$-\sum_{i=1}^{N}\phi_i\left(\Delta_i - \sum_{m=1}^{\infty} A_m \sin\frac{2m\pi c_i}{b}\right)$$

$$-\sum_{i=1}^{N}\psi_i\left(\Gamma_i - \sum_{n=0}^{\infty} B_n \cos\frac{2n\pi c_i}{b}\right) \qquad (16)$$

If the function f is minimized with respect to each of the parameters A_m, B_n, Δ_i, and Γ_i in turn, the following expressions can be obtained:

$$\frac{\partial f}{\partial A_m} = 2FK_m'A_m - F_{xy}\pi^2 mB_m - \epsilon m(-1)^m + \sum_{i=1}^{N} \phi_i \sin \frac{2m\pi c_i}{b}$$

$$= 0 \qquad (m = 1,2,\ldots\infty) \tag{17}$$

$$\frac{\partial f}{\partial B_n} = 2FK_n'\left(1 + \delta_{0n}\right)B_n - F_{xy}\pi^2 nA_n - \nu(-1)^n$$

$$+ \sum_{i=1}^{N} \psi_i \cos \frac{2n\pi c_i}{b} = 0 \qquad (n = 0,1,2,\ldots\infty) \tag{18}$$

$$\frac{\partial f}{\partial \Delta_i} = 2F\gamma_i\left(2\Delta_i\right) - \phi_i$$

$$= 0 \qquad (i = 1,2,\ldots N) \tag{19}$$

$$\frac{\partial f}{\partial \Gamma_i} = 2F\gamma_i\left(2\Gamma_i\right) - \psi_i$$

$$= 0 \qquad (i = 1,2,\ldots N) \tag{20}$$

Equation (18) can be separated into the following parts:

$$4FB_0 - \nu + \sum_{i=1}^{N} \psi_i = 0 \qquad \text{(corresponding to } n = 0) \tag{21a}$$

$$2FK_m'B_m - F_{xy}\pi^2 mA_m - \nu(-1)^m + \sum_{i=1}^{\infty} \psi_i \cos \frac{2m\pi c_i}{b} = 0 \tag{21b}$$

$$(m = 1,2,\ldots\infty)$$

Solving equations (17) and (21) together and using the notations $\nu' = \frac{\nu}{2F}$, $\epsilon' = \frac{\epsilon}{2F}$, $\phi_i' = \frac{\phi_i}{2F}$, and $\psi_i' = \frac{\psi_i}{2F}$ result in the following expressions for the A's and B's:

$$A_m = R_m\left[\epsilon' m(-1)^m - \sum_{i=1}^{N} \phi_i' \sin\frac{2m\pi c_i}{b}\right] + S_m\left[\nu'(-1)^m - \sum_{i=1}^{N} \psi_i' \cos\frac{2m\pi c_i}{b}\right] \qquad (m = 1,2,\ldots\infty) \qquad (22)$$

$$B_m = R_m\left[\nu'(-1)^m - \sum_{i=1}^{N} \psi_i' \cos\frac{2m\pi c_i}{b}\right] + S_m\left[\epsilon' m(-1)^m - \sum_{i=1}^{N} \phi_i' \sin\frac{2m\pi c_i}{b}\right] \qquad (m = 1,2,\ldots\infty) \qquad (23a)$$

$$B_0 = \frac{1}{2}\nu' - \frac{1}{2}\sum_{i=1}^{N} \psi_i' \qquad (23b)$$

where

$$R_m = \frac{K_m'}{\left(K_m'\right)^2 - (mQ)^2}$$

$$S_m = \frac{mQ}{\left(K_m'\right)^2 - (mQ)^2}$$

$$Q = \frac{F_{xy}\pi^2}{2F} = 4\beta^3 k$$

If equations (19) and (20) are substituted in equations (14), the following expressions result:

$$\frac{1}{2\gamma_i}\phi_i' - \sum_{m=1}^{\infty} A_m \sin\frac{2m\pi c_i}{b} = 0 \qquad (i = 1,2,\ldots N) \qquad (24a)$$

$$\frac{1}{2\gamma_i}\psi_i' - \sum_{n=0}^{\infty} B_n \cos\frac{2n\pi c_i}{b} = 0 \qquad (i = 1,2,\ldots N) \qquad (24b)$$

If A_m and B_n as given by equations (22) and (23) are put into equations (12) and (24), the following set of simultaneous equations is obtained with ν', ϵ', ϕ_i', and ψ_i' as the variables:

$$\nu'R + \epsilon'S + \sum_{i=1}^{N} \phi_i'\left(-P_i\right) + \sum_{i=1}^{N} \psi_i'\left(-H_i\right) = 0 \tag{25a}$$

$$\nu'S + \epsilon'G + \sum_{i=1}^{N} \phi_i'\left(-L_i\right) + \sum_{i=1}^{N} \psi_i'\left(-T_i\right) = 0 \tag{25b}$$

$$\nu'\left(-P_j\right) + \epsilon'\left(-L_j\right) + \sum_{i=1}^{N} \phi_i'\left(X_{ij}\right) + \phi_j' \frac{1}{2\gamma_j} + \sum_{i=1}^{N} \psi_i'\left(W_{ij}\right) = 0 \tag{25c}$$

$$(j = 1,2,\ldots N)$$

$$\nu'\left(-H_j\right) + \epsilon'\left(-T_j\right) + \sum_{i=1}^{N} \phi_i'\left(W_{ij}\right) + \sum_{i=1}^{N} \psi_i'\left(Z_{ij}\right) + \psi_j'\left(\frac{1}{2\gamma_j}\right) = 0 \tag{25d}$$

$$(j = 1,2,\ldots N)$$

where

$$R = \frac{1}{2} + \sum_{m=1}^{\infty} R_m$$

$$G = \sum_{m=1}^{\infty} m^2 R_m$$

$$H_j = \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^m R_m \cos \frac{2m\pi c_j}{b}$$

$$L_j = \sum_{m=1}^{\infty} m(-1)^m R_m \sin \frac{2m\pi c_j}{b}$$

$$X_{ij} = \sum_{m=1}^{\infty} R_m \sin \frac{2m\pi c_i}{b} \sin \frac{2m\pi c_j}{b} = X_{ji}$$

$$Z_{ij} = \frac{1}{2} + \sum_{m=1}^{\infty} R_m \cos \frac{2m\pi c_i}{b} \cos \frac{2m\pi c_j}{b} = Z_{ji}$$

$$S = \sum_{m=1}^{\infty} mS_m$$

$$P_j = \sum_{m=1}^{\infty} (-1)^m S_m \sin \frac{2m\pi c_j}{b}$$

$$T_j = \sum_{m=1}^{\infty} m(-1)^m S_m \cos \frac{2m\pi c_j}{b}$$

$$W_{ij} = \sum_{m=1}^{\infty} S_m \cos \frac{2m\pi c_i}{b} \sin \frac{2m\pi c_j}{b}$$

There are $2N + 2$ equations in the preceding system of simultaneous equations and there are $2N + 2$ unknowns, namely, ν', ϵ', ϕ_1', ϕ_2', $\ldots\phi_N'$, ψ_1', ψ_2', $\ldots$ and ψ_N'. In order to ensure a nonvanishing solution for these Lagrangian multipliers, the following determinant must be zero:

	ν'	ϵ'	ϕ_1'	ψ_1'	ϕ_2'	ψ_2'	. . .	ϕ_N'	ψ_N'
Equation (25a)	R	S	$-P_1$	$-H_1$	$-P_2$	$-H_2$	. . .	$-P_N$	$-H_N$
Equation (25b)	S	G	$-L_1$	$-T_1$	$-L_2$	$-T_2$	. . .	$-L_N$	$-T_N$
Equation (25c), j=1	$-P_1$	$-L_1$	$X_{11}+\frac{1}{2\gamma_1}$	W_{11}	X_{21}	W_{21}	. . .	X_{N1}	W_{N1}
Equation (25d), j=1	$-H_1$	$-T_1$	W_{11}	$Z_{11}+\frac{1}{2\gamma_1}$	W_{12}	Z_{21}	. . .	W_{1N}	Z_{N1}
Equation (25c), j=2	$-P_2$	$-L_2$	X_{12}	W_{12}	$X_{22}+\frac{1}{2\gamma_2}$	W_{22}	. . .	X_{N2}	W_{N2}
Equation (25d), j=2	$-H_2$	$-T_2$	W_{21}	Z_{12}	W_{22}	$Z_{22}+\frac{1}{2\gamma_2}$	. . .	W_{2N}	Z_{N2}
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
Equation (25c), j=N	$-P_N$	$-L_N$	X_{1N}	W_{1N}	X_{2N}	W_{2N}	. . .	$X_{NN}+\frac{1}{2\gamma_N}$	W_{NN}
Equation (25d), j=N	$-H_N$	$-T_N$	W_{N1}	Z_{1N}	W_{N2}	Z_{2N}	. . .	W_{NN}	$Z_{NN}+\frac{1}{2\gamma_N}$

$\equiv 0$ (26)

Determinant (26) is of the $2N + 2$ order; it can be greatly simplified by using the numbering system shown in figure 9 for the longitudinal stiffeners, which are assumed to have equal stiffness γ_o and are equally spaced across the width of the plate. In this system, the first stiffener, if any, is the central stiffener; the numbers 2, 4, . . . denote the stiffeners on the positive y-side of the plate, and the numbers 3, 5, . . . indicate the stiffeners on the negative y-side, numbering out from the center in each case. With this numbering system and where $p = 1, 2, 3, \ldots$, the following relations follow immediately:

$$c_{2p} = -c_{2p+1} \qquad Z_{11} = R$$

$$H_{2p} = H_{2p+1} \qquad P_{2p} = -P_{2p+1}$$

$$T_{2p} = T_{2p+1} \qquad L_{2p} = -L_{2p+1}$$

$$W_{2p,i} = W_{2p+1,i} \qquad X_{2p,i} = -X_{2p+1,i}$$

$$Z_{2p,i} = Z_{2p+1,i} \qquad W_{i,2p} = -W_{i,2p+1}$$

$$P_1 = L_1 = X_{1,i} = W_{i,1} = 0$$

With these relations, the determinant can be simplified by means of operations of the following type: First, columns under ϕ_{2p}' are added to columns under ϕ_{2p+1}'; then rows corresponding to equation (25c) with $j = 2p + 1$ are subtracted from rows under equation (25c) with $j = 2p$. In this way, all the columns under ϕ_{2p+1}' are eliminated in determinant (26). Similar operations can be used to eliminate all columns under Ψ_{2p+1}'. After dividing some rows for convenience by 2, the resulting determinant becomes

$$\begin{vmatrix} R & S & -H_1 & -P_2 & -H_2 & -P_4 & -H_4 & \cdots \\ S & G & -T_1 & -L_2 & -T_2 & -L_4 & -T_4 & \cdots \\ -H_1 & -T_1 & R+\frac{1}{2\gamma_o} & W_{12} & Z_{21} & W_{14} & Z_{41} & \cdots \\ -P_2 & -L_2 & W_{12} & X_{22}+\frac{1}{4\gamma_o} & W_{22} & X_{42} & W_{42} & \cdots \\ -H_2 & -T_2 & Z_{12} & W_{22} & Z_{22}+\frac{1}{4\gamma_o} & W_{24} & Z_{42} & \cdots \\ -P_4 & -L_4 & W_{14} & X_{24} & W_{24} & X_{44}+\frac{1}{4\gamma_o} & W_{44} & \cdots \\ -H_4 & -T_4 & Z_{14} & W_{42} & Z_{24} & W_{44} & Z_{44}+\frac{1}{4\gamma_o} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \quad (27)$$

Determinant (27) is now of the $N + 2$ order. For the particular cases of one and two longitudinal stiffeners, the corresponding determinants are as follows:

For one stiffener

$$\begin{vmatrix} R & S & -H_1 \\ S & G & -T_1 \\ -H_1 & -T_1 & R+\frac{1}{2\gamma_o} \end{vmatrix} = 0 \quad (28)$$

For two stiffeners

$$\begin{vmatrix} R & S & -P_2 & -H_2 \\ S & G & -L_2 & -T_2 \\ -P_2 & -L_2 & X_{22}+\frac{1}{4\gamma_o} & W_{22} \\ -H_2 & -T_2 & W_{22} & Z_{22}+\frac{1}{4\gamma_o} \end{vmatrix} = 0 \quad (29)$$

In determinants (27) to (29), each element is an infinite series consisting of functions of k and β. For each of the determinants, γ_o can be computed for various values of β at a prescribed value of k. A maximum value of γ_o for this prescribed k can be obtained by plotting γ_o against β. A number of these maximum values of γ_o corresponding to various values of this prescribed k can be obtained. Instead of plotting k against γ_o, k was plotted against γ, as was done for the case of simply supported edges, where $\gamma = J\gamma_o$ and J equals the number of bays into which the plate is divided by the stiffeners. Thus, $J = 2$ for one stiffener and $J = 3$ for two stiffeners. The advantage of such a plot is that the various curves for different numbers of stiffeners become almost coincident. (See fig. 3.)

Large Number of Stiffeners

In the case of a very large number of stiffeners of equal stiffness and equally spaced across the width of the plate, the bending energy stored in the stiffeners may be expressed, just as in the case of simply supported edges, as follows:

$$V_s = \frac{1}{2}\frac{EI}{d}\int_{-b/2}^{b/2}\int_0^{\lambda}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\,dy$$

If, now, the same deflection function as given by equation (11) for the plate is used for the stiffeners, the following expression is obtained for the bending energy stored in the stiffeners:

$$V_s = F\gamma\left[\sum_{m=1}^{\infty} A_m^2 + \sum_{n=0}^{\infty} B_n^2\left(1 + \delta_{0n}\right)\right] \qquad (30)$$

There is no change of expressions in V_p and T. Of course, since a separate deflection function for the stiffeners is no longer used, there is no need for the restraining conditions of equations (14). The function to be minimized is

$$f = V_p + V_s - T - \nu\sum_{n=0}^{\infty}(-1)^n B_n - \epsilon\sum_{m=1}^{\infty} m(-1)^m A_m \qquad (31)$$

If the same steps are followed as before, the following determinantal equation is found to be the condition for the existence of nonvanishing solutions for the Lagrangian multipliers:

$$\begin{vmatrix} \sum_{m=1}^{\infty} m^2 R_m' & \sum_{m=1}^{\infty} m S_m' \\ \sum_{m=1}^{\infty} m S_m' & \frac{1}{2(1+\gamma)} + \sum_{m=1}^{\infty} R_m' \end{vmatrix} = 0 \qquad (32)$$

where

$$R_m' = \frac{K_m' + \gamma}{\left(K_m' + \gamma\right)^2 - (mQ)^2}$$

$$S_m' = \frac{mQ}{\left(K_m' + \gamma\right)^2 - (mQ)^2}$$

In determinant (32), each element contains an infinite series of functions of k, γ, and β. For prescribed values of both γ and β, several values are assigned for k and the corresponding values of the determinant can be determined. If the values of the determinant are plotted against k, one value of k can be found which makes the value of the determinant vanish. This particular value of k is called k_o. Now, if the value of β is changed (while the value of γ remains the same), the corresponding value of k_o is also changed. There exists a certain β which makes the value of k_o a minimum. This minimum value of k_o is the critical shear buckling coefficient k corresponding to the prescribed value of γ. In a similar manner, other critical shear buckling coefficients can be determined for other prescribed values of γ. Finally, a curve can be obtained with k plotted against γ. This curve is presented in figures 3 and 8. In figure 8, the exact solution for an orthotropic plate from reference 2 was also plotted for comparison.

From the results obtained in the case of simply supported edges, it is believed that the curve obtained for a large number of identical equally spaced stiffeners represents the solution for all cases of three or more stiffeners.

DISCUSSION OF THEORETICAL RESULTS

In the foregoing analysis it is seen that for both simply supported and clamped edges a determinant is obtained for the determination of the shear buckling coefficient k. In the case of simply supported edges, the usual energy method was used and the determinant has been solved by first, second, and third approximations. They converge very fast, however, as may be judged from the results of the solution for one stiffener. (See fig. 7.) Also, figure 8 shows that the second-approximation solution for a large number of stiffeners is sufficient to give excellent agreement with results obtained from reference 2. In the case of clamped edges the Lagrangian multiplier method was used and the solution of the determinant is exact. If enough terms in each infinite series are taken, the solution can be made to any desired accuracy.

Curves of the stiffness factor γ plotted against the shear buckling coefficient k for one centrally located stiffener, two identical equally spaced stiffeners, and a large number of identical equally spaced stiffeners are shown in figure 3 for both simply supported edges and clamped edges. The curve for a large number of identical equally spaced stiffeners represents the solution for all cases of three or more stiffeners. Since the curves of figure 3 do not depart from one another by more than about 2 or 3 percent over the range shown, practicability would dictate the use of the curve for a large number of identical equally spaced stiffeners to predict the buckling load for a plate with any number of stiffeners, provided the buckling coefficient so obtained is not higher than could be obtained by replacing the stiffeners by simple supports. On the basis of the criterion just stated, it is apparent from figure 3 that in the case of a simply supported plate reinforced with a single stiffener no further increase in the shear buckling coefficient k can be obtained by increasing γ beyond about 2000. Similarly, it is seen that for the case of a simply supported plate reinforced by two identical equally spaced stiffeners no further increase in the shear buckling coefficient k can be obtained by increasing γ beyond about 45,000.

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3. Kromm, A.: The Limit of Stability of a Curved Plate Strip under Shear and Axial Stresses. NACA TM No. 898, 1939.

4. Budiansky, Bernard, Seide, Paul, and Weinberger, Robert A.: The Buckling of a Column on Equally Spaced Deflectional and Rotational Springs. NACA TN No. 1519, 1947.

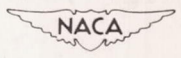
5. Budiansky, Bernard, and Hu, P. C.: The Lagrangian Multiplier Method of Finding Upper and Lower Limits to Critical Stresses of Clamped Plates. NACA TN No. 1103, 1946.

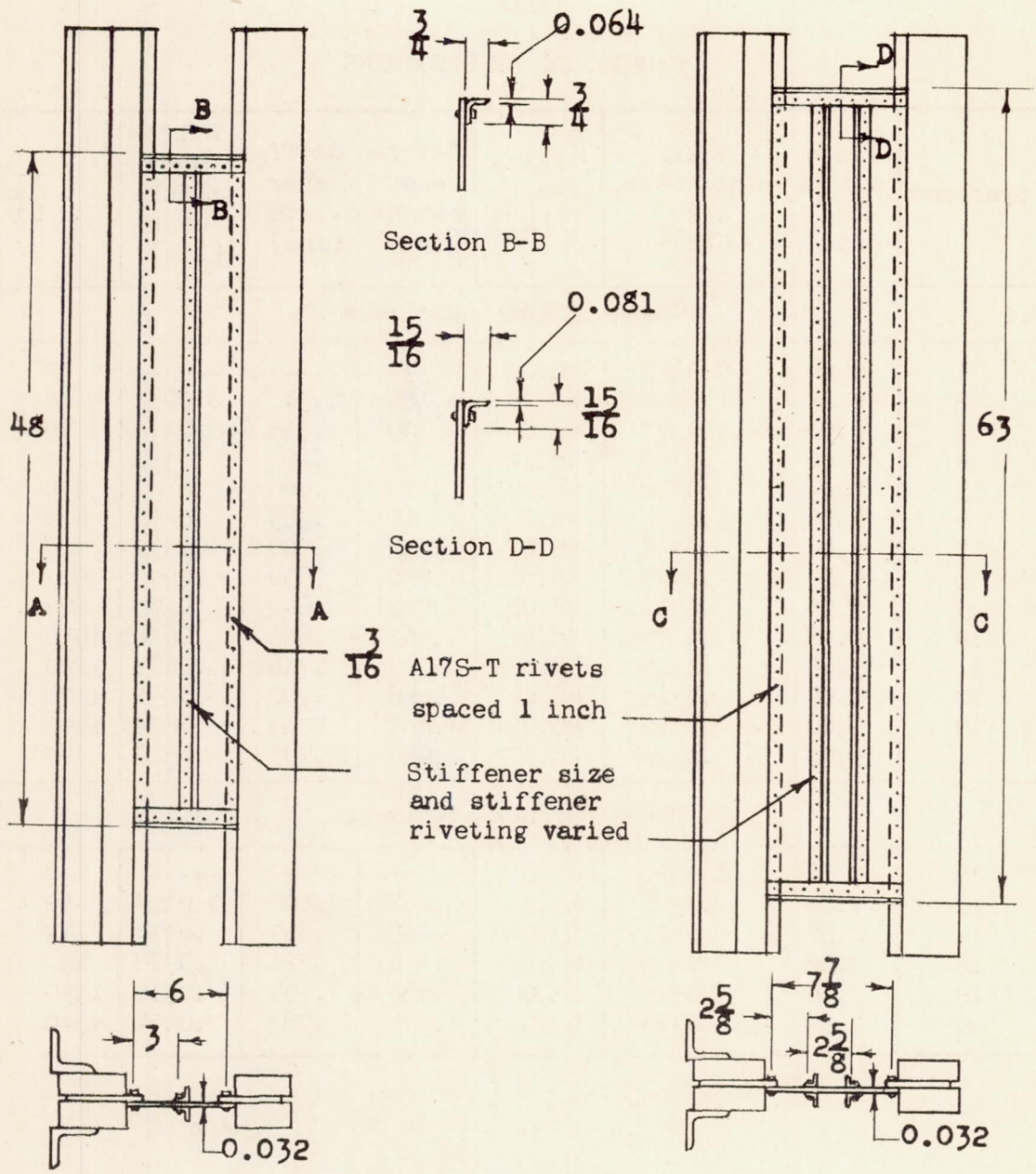
6. Budiansky, Bernard, Hu, P. C., and Connor, Robert W.: Notes on the Lagrangian Multiplier Method in Elastic-Stability Analysis. NACA TN No. 1558, 1948.

TABLE 1

DIMENSIONS OF SPECIMENS

Specimen	Plate width, b (in.)	Plate thickness, t (in.)	Plate length, (in.)	Stiffener height (in.)	Stiffener width (in.)	Stiffener thickness (in.)	$\gamma = \frac{EI}{Dd}$
One-stiffener specimens							
1	6.00	0.0340	48.00	-----	-----	------	0
2	6.04	.0334	47.98	0.185	0.31	0.0281	16.5
3	6.02	.0337	48.05	.300	.315	.0376	78.4
4	6.00	.0330	48.12	.305	.290	.0331	153
5	6.01	.0325	48.09	.420	.460	.0391	234
6	5.99	.0334	48.00	.510	.385	.0494	477
7	6.00	.0326	48.07	.550	.500	.0391	503
8	6.00	.0327	48.06	.550	.650	.0492	633
9	5.96	.0313	48.00	.622	.642	.0395	824
10	5.98	.0316	48.02	.581	.653	.0626	1050
11	5.96	.0315	48.04	.623	.718	.0621	1290
12	6.00	.0315	48.05	.668	.700	.0630	1570
13	6.00	.0312	48.03	.665	.715	.0622	1590
14	6.06	.0330	48.02	.985	.795	.0618	4180
Two-stiffener specimens							
15	7.88	0.0326	63.0	-----	-----	------	0
16	7.88	.0326	63.0	0.248	0.505	0.0327	53.4
17	7.88	.0321	63.0	.348	.649	.0334	147
18	7.88	.0326	63.0	.550	.721	.0328	484
19	7.88	.0325	63.0	.685	.794	.0510	1390
20	7.88	.0325	63.0	.944	.783	.0634	4540





Section A-A

(a) Specimen with one stiffener.

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Section C-C

(b) Specimen with two stiffeners.

Figure 1.- Test specimens. (All dimensions are in inches.)

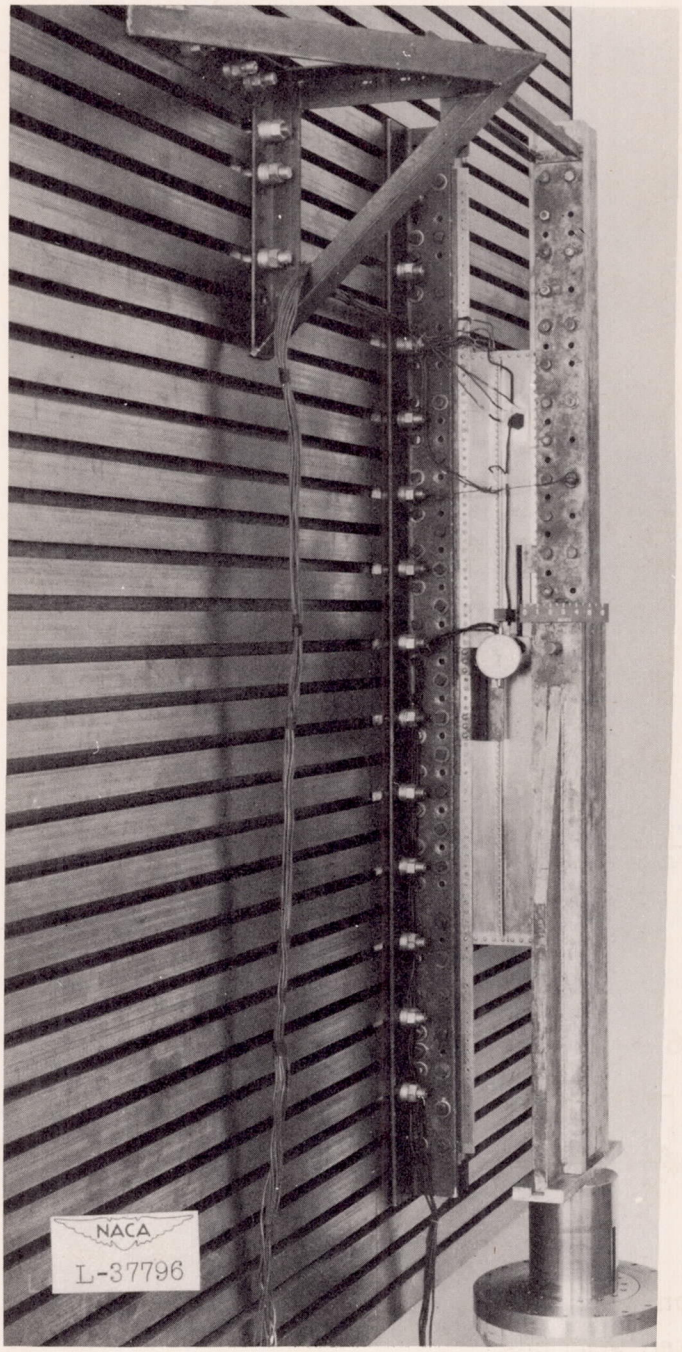


Figure 2.- Photograph of test setup.

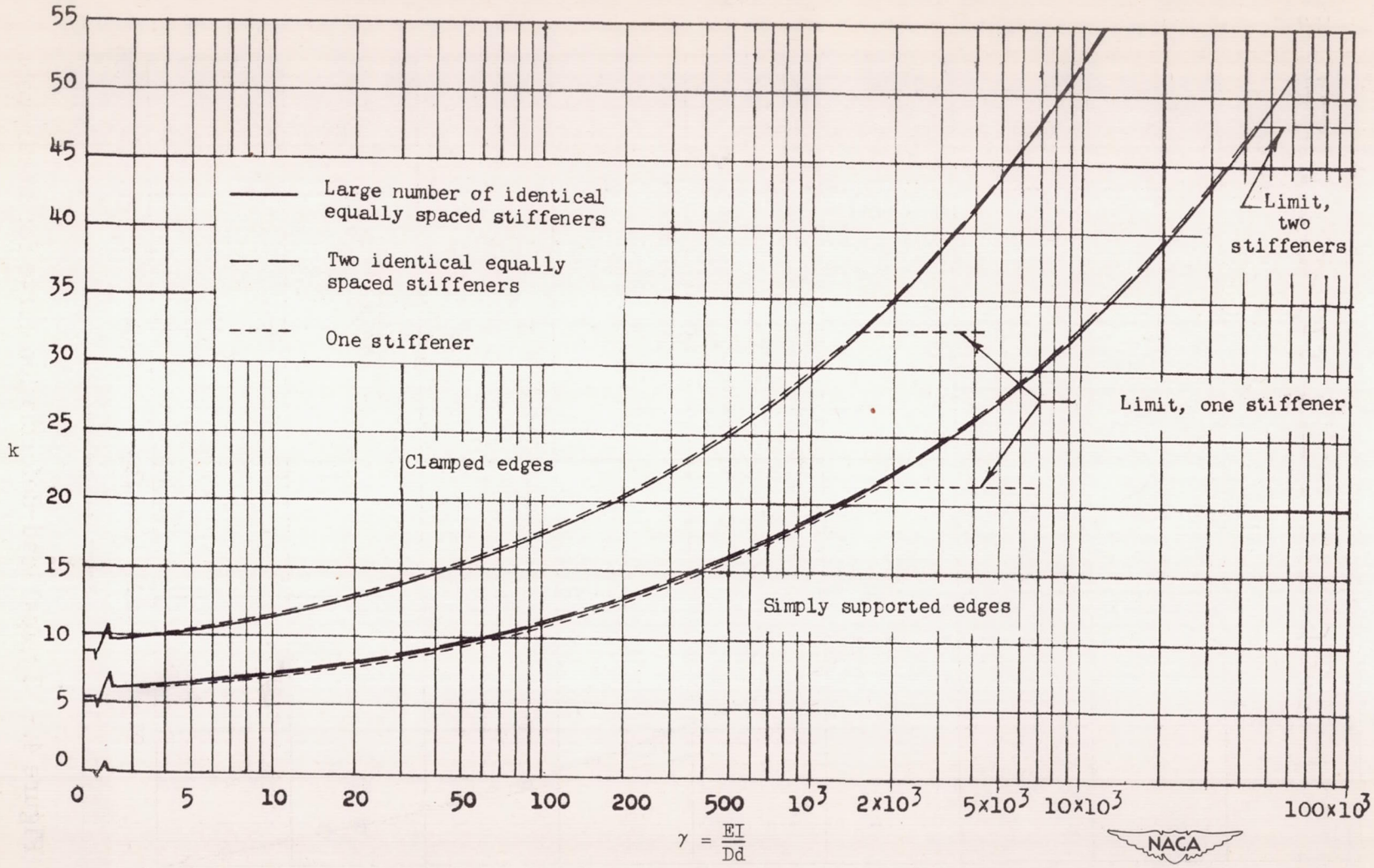


Figure 3.- Variation of shear buckling coefficient with rib stiffness.

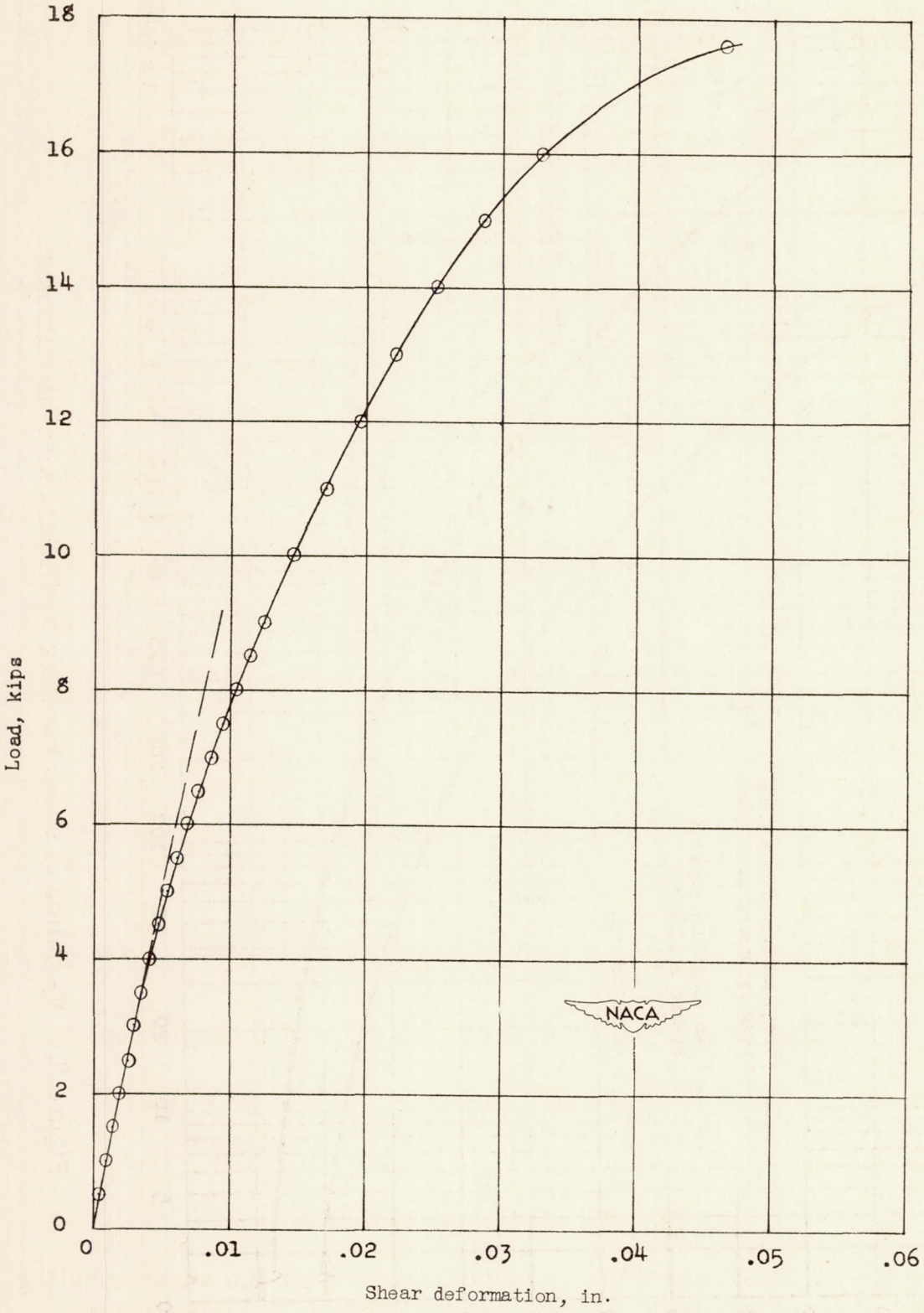
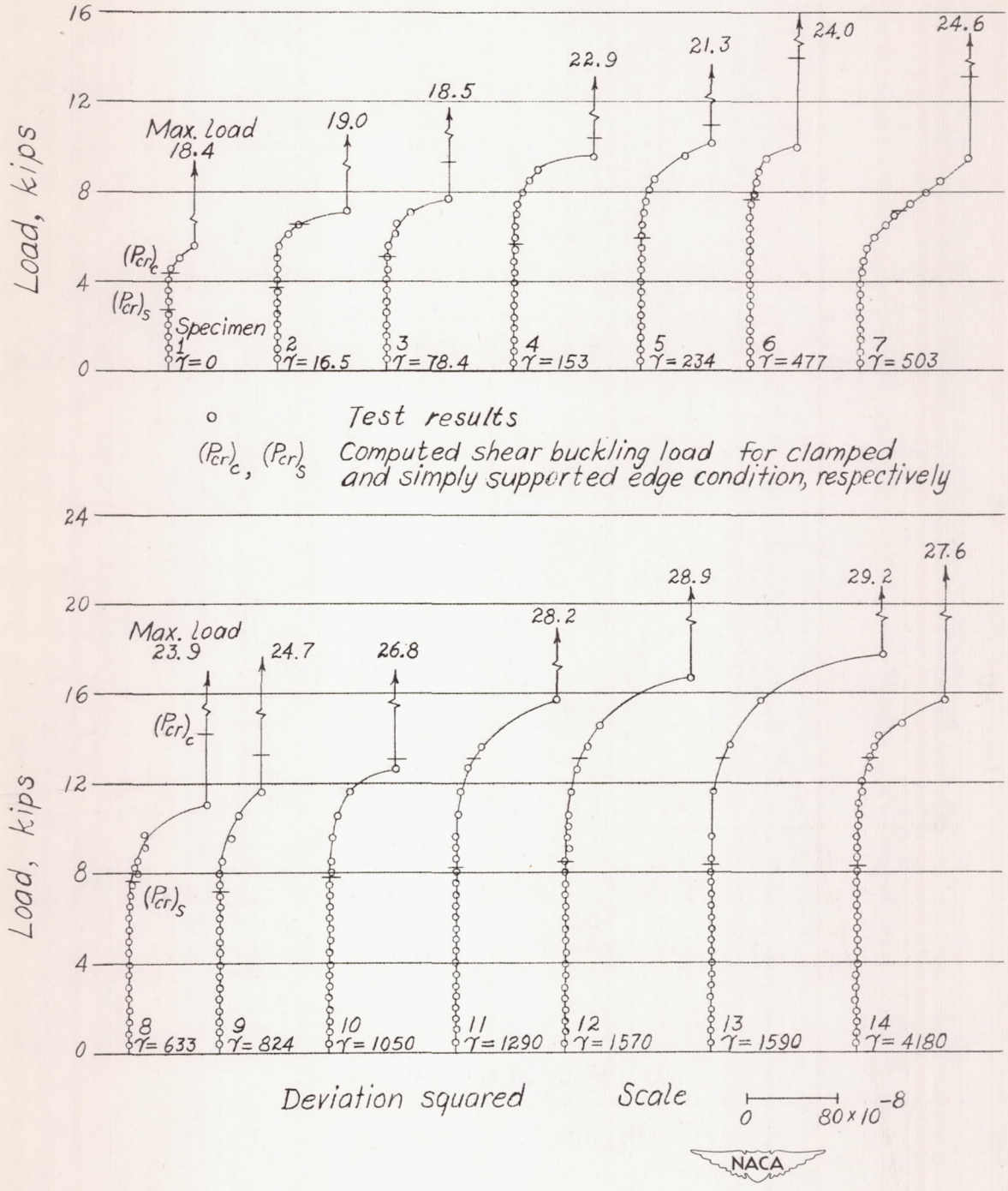
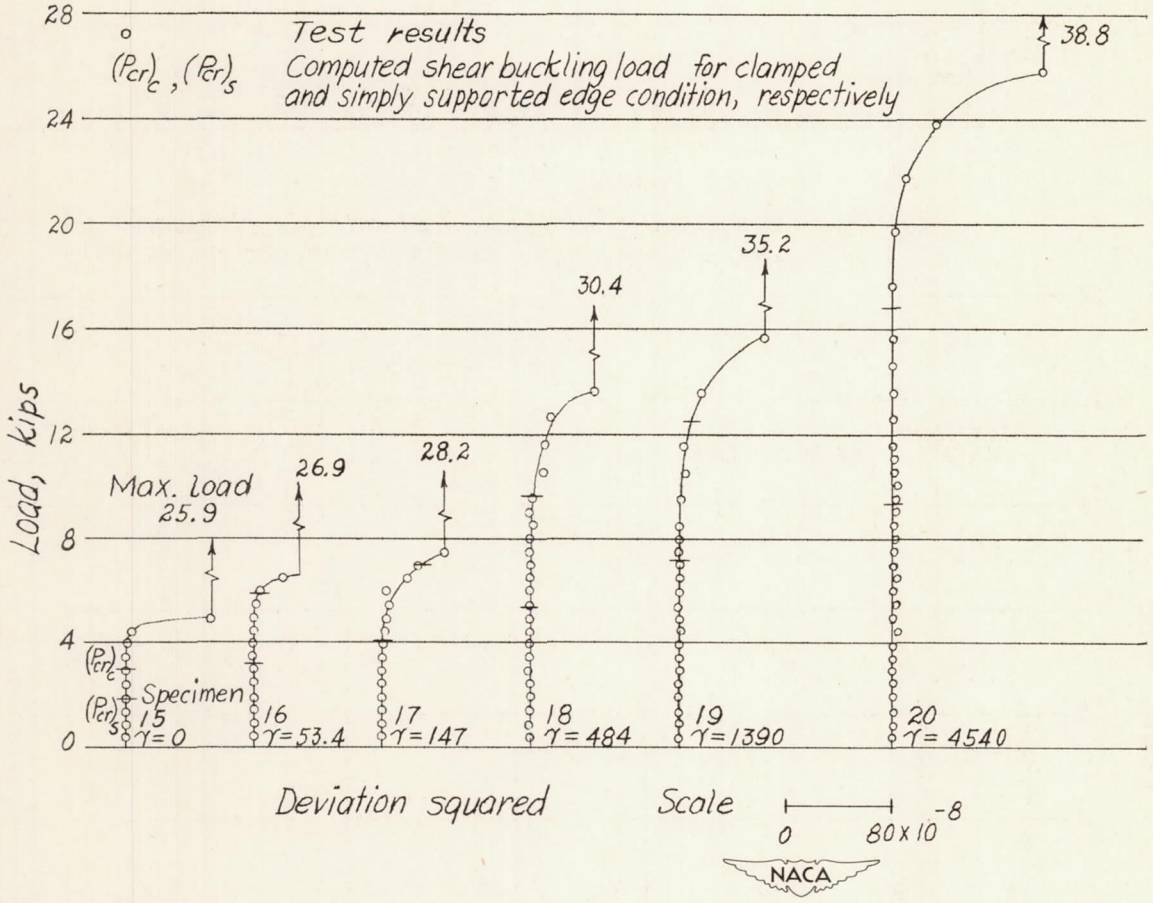


Figure 4.- Typical load-deformation curve for shear webs.



(a) Webs with one stiffener.

Figure 5.- Comparison of test results and theory on the shear buckling load of long flat plates with longitudinal stiffeners, identical and equally spaced.



(b) Webs of two stiffeners.

Figure 5.- Concluded.

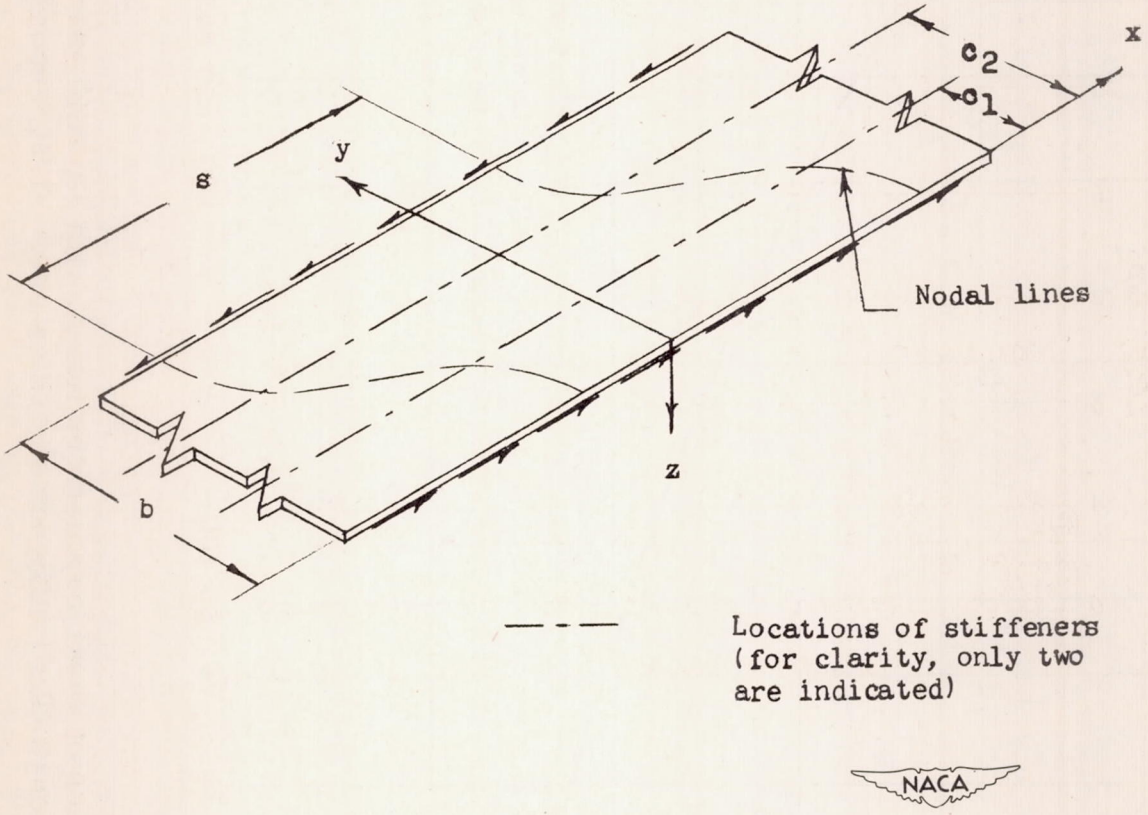


Figure 6.- Coordinate system used in theory for simply supported plates.

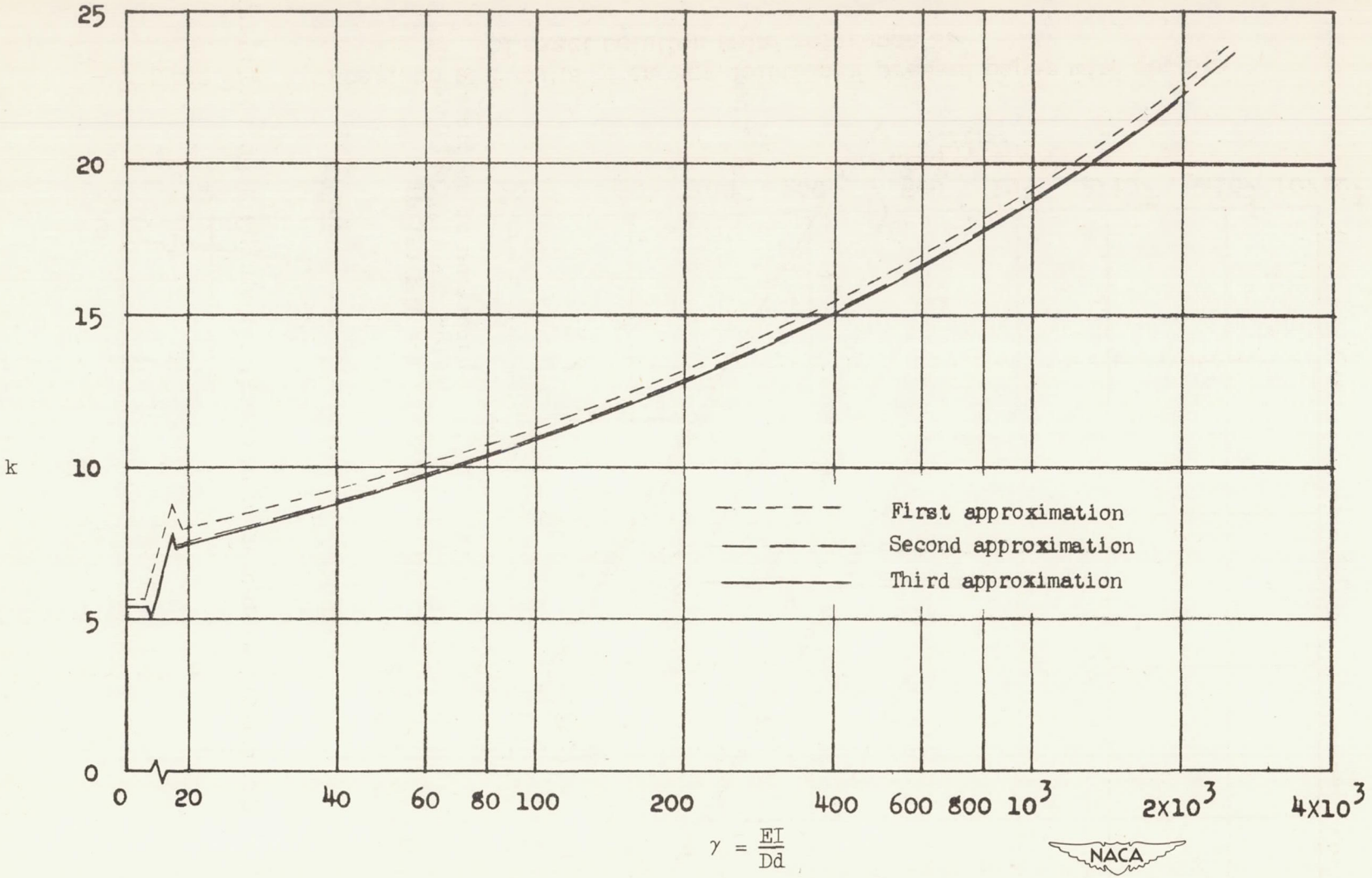


Figure 7.- Variation of shear buckling coefficient with rib stiffness for simply supported plate with one longitudinal stiffener on center line. First, second, and third approximations.

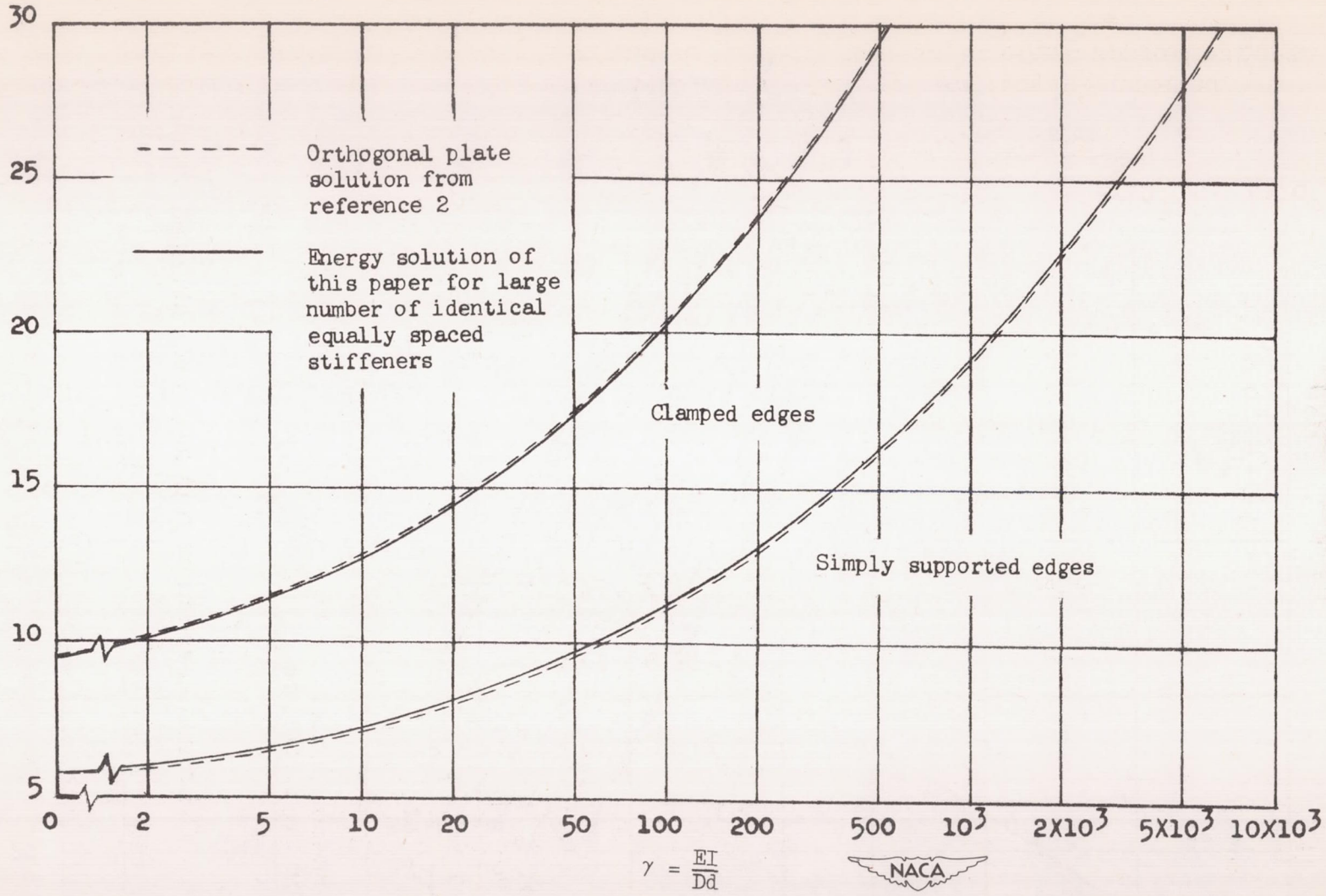


Figure 8.- Comparison of results of energy solution of present paper with results of exact solution from reference 2.

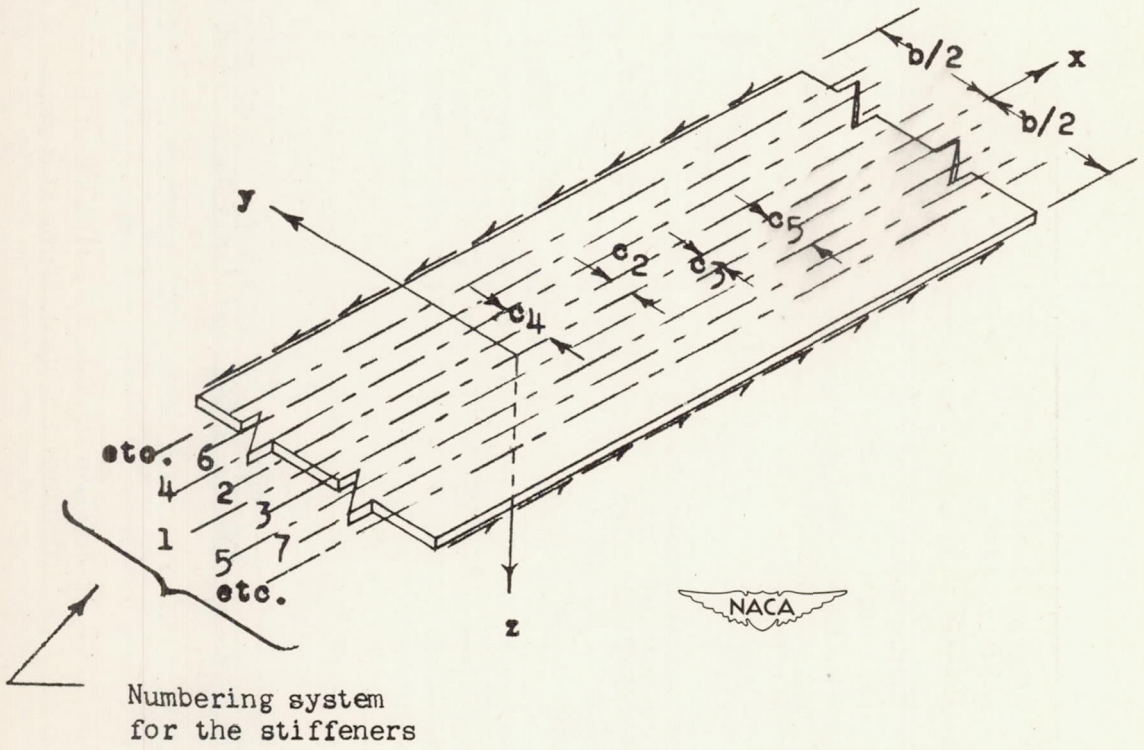


Figure 9.- Coordinate system used in theory for clamped plates.